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CONTINUITY OF THE STRAIGHT LINE PATH FOR CONVEX SETS

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ABSTRACT
If A and B are closed nonempty sets in a locally convex space, the straight line path from A to B is defined by the formula $\phi(\alpha) = \mathrm{cl}(\alpha A + (1-\alpha)B)$, $0 \leqq \alpha \leqq 1$. If A and B are convex, then continuity of the path with respect to the Hausdorff uniform topology is necessary for both connectedness and path connectedness of A to B within the convex sets so topologized. We also produce internal necessary and sufficient conditions for continuity of the path between pairs of convex sets.

Let X be a normed linear space with closed unit ball U_0, and let $C(X)$ denote the collection of nonempty closed convex subsets of X. *Hausdorff distance* is defined on $C(X)$ by means of the familiar formula

$$H(A,B) = \inf\{\varepsilon : A + \varepsilon U_0 \supset B \quad \text{and} \quad B + \varepsilon U_0 \supset A\}.$$

Hausdorff distance so defined determines an infinite valued metric on $C(X)$. If X is now a locally convex topological vector space, there is a natural way to generalize this topology to $C(X)$ in this broader context. Notice that the natural uniformity for H in the normed case has as a base for its entourages all sets of the form

$$\left\{(A,B) : A + \frac{1}{n} U_0 \supset B \quad \text{and} \quad B + \frac{1}{n} U_0 \supset A\right\}.$$

With this in mind, we define the *Hausdorff uniform topology* τ_H on $C(X)$ in the more general setting to be the topology determined by the uniformity on $C(X)$ with a base whose typical entourage is of the form

$$\Omega_V = \{(A,B) : A + V \supset B \quad \text{and} \quad B + V \supset A\}$$

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where V runs over a given local base for the topology of X at the origin θ. Of course, the topology does not depend on the choice of the local base. In the sequel, we denote the base consisting of the balanced open convex neighborhoods of θ by $\mathcal{A}$. A basic reference for the Hausdorff uniform topology is [1], where the important theorem of Hormander [6] and its consequences are presented: the closed and bounded convex subsets of a Hausdorff locally convex space so topologized can be imbedded as a cone in a locally convex space, where scalar multiplication is defined in the usual way, and addition is the closure of the usual Minkowski sum (the Minkowski sum of closed convex sets need not be closed).

It is, of course, possible to take convex combinations of the elements of a cone, and in this way, we can speak of convex combinations of closed convex sets. In particular, we can speak of the line segment joining a pair of closed convex sets. One of the richest parts of convexity theory in Euclidean space involves the volumes of such combinations and the related concept of mixed volumes (cf. [4] and [11]). Perhaps the main result here is the Brünn–Minkowski inequality: if A and B are compact convex sets in n-dimensional Euclidean space and μ denotes n-dimensional Lebesgue measure, then for each $0 \leqq \alpha \leqq 1$,

$$[\mu(\alpha A + (1-\alpha)B)]^{1/n} \geqq \alpha[\mu(A)]^{1/n} + (1-\alpha)[\mu(B)]^{1/n}.$$

Put more simply, $\alpha \to [\mu(\alpha A + (1-\alpha)B)]^{1/n}$ is a concave function of α.

We return again to the setting of a locally convex topological vector space X. In this note we consider the continuity of the *straight line path* between two elements A and B of $C(X)$, defined by

$$\phi(\alpha) = \mathrm{cl}\,(\alpha A + (1-\alpha)B), \qquad 0 \leqq \alpha \leqq 1,$$

where $C(X)$ is equipped with the above topology. That ϕ need not be continuous even when $X = R$ is obvious: if $A = R$ and $B = \{0\}$, then

$$\phi(\alpha) = \begin{cases} \{0\} & \text{if } \alpha = 0 \\ R & \text{otherwise} \end{cases}$$

is not continuous at the origin. From this example, one might guess that problems can only occur at $\alpha = 0$ and $\alpha = 1$, but this is not the case. In the plane, if $A = \{(x, y) : y \geqq x^2\}$ and $B = \{(0,0)\}$, then

$$\phi(\alpha) = \begin{cases} \{(x, y) : y \geqq x^2/\alpha\} & \text{if } \alpha \neq 0, \\ \{(0,0)\} & \text{if } \alpha = 0. \end{cases}$$

Now if $\beta \neq \beta'$, the Hausdorff distance between the parabolas $y = \beta x^2$ and $y = \beta' x^2$ with respect to the usual norm on R^2 is infinite; so, ϕ is everywhere discontinuous.

A priori, we might expect that if the straight line path from A to B fails to be continuous, we might still be able to find a different continuous path from A to B in $\langle C(X), \tau_H \rangle$. We show that this can never happen, although it certainly can if A and B are not assumed convex.

EXAMPLE. In R^2 with the usual norm, let

$$A = \{x, y): \text{either } x = 0 \text{ or } y = 0\}$$

and

$$B = A + U_0 = \{(x, y): \text{either } |x| \leqq 1 \text{ or } |y| \leqq 1\}.$$

Although neither set is convex, both are *starshaped* [12] with respect to the origin, i.e., $(0,0)$ sees each point of A (resp. B) via A (resp. B). Clearly,

$$\psi(\alpha) = A + \alpha U_0$$

is a continuous path from A to B in the closed subsets of the plane with the Hausdorff metric. However, the straight line path ϕ is not, because if $0 < \alpha < 1$ we have

$$\phi(\alpha) = \alpha A + (1-\alpha)A + (1-\alpha)U_0 \supset \alpha A + (1-\alpha)A = A + A = R^2.$$

Path connectedness (in fact, connectedness) in the hyperspace of convex sets turns out to be characterized by a boundedness condition.

THEOREM 1. *Let X be locally convex space, and let $\mathcal{A}$ be a local base at the origin of open balanced convex sets. Then for each pair of closed convex sets in $C(X)$, the following are equivalent*:

(i) *For each $V \in \mathcal{A}$, there exists $n \in Z^+$ with both $A + nV \supset B$ and $B + nV \supset A$*;

(ii) *The map $\phi : [0,1] \to \langle C(X), \tau_H \rangle$ defined by $\phi(\alpha) = \mathrm{cl}(\alpha A + (1-\alpha)B)$ is continuous*;

(iii) *The map $\phi : [0,1] \to \langle C(X), \tau_H \rangle$ defined by $\phi(\alpha) = \mathrm{cl}(\alpha A + (1-\alpha)B)$ is continuous at $\alpha = 0$ and $\alpha = 1$*;

(iv) *A and B are path connected in $\langle C(X), \tau_H \rangle$*;

(v) *A and B are connected in $\langle C(X), \tau_H \rangle$*.

PROOF. (i)$\to$(ii). Let $W \in \mathcal{A}$ be arbitrary. We produce $\delta > 0$ such that whenever $|\alpha - \alpha'| < \delta$, then $(\phi(\alpha), \phi(\alpha')) \in \Omega_W$. Choose $V \in \mathcal{A}$ for which

$V+V\subset W$, and choose $n\in Z^+$ for which both $A+nV\supset B$ and $B+nV\supset A$. Since V is balanced, for each $\varepsilon\in(0,1/n]$ we have $\varepsilon A+V\supset\varepsilon B$ and $\varepsilon B+V\supset \varepsilon A$. Suppose $0<|\alpha-\alpha'|<1/n$. First, if $\alpha>\alpha'$, we have

$$\begin{aligned}\alpha A+(1-\alpha)B+W &\supset \alpha' A+(\alpha-\alpha')A+V+(1-\alpha)B+V\\ &\supset \alpha' A+(\alpha-\alpha')B+(1-\alpha)B+V\\ &= \alpha' A+(1-\alpha')B+V\\ &\supset \mathrm{cl}\,(\alpha' A+(1-\alpha')B).\end{aligned}$$

Otherwise, $\alpha<\alpha'$ and

$$\begin{aligned}\alpha A+(1-\alpha)B+W &\supset \alpha A+(1-\alpha')B+(\alpha'-\alpha)B+V+V\\ &\supset \alpha A+(1-\alpha')B+(\alpha'-\alpha)A+V\\ &\supset \mathrm{cl}\,(\alpha' A+(1-\alpha')B).\end{aligned}$$

Thus in either case, $\phi(\alpha)+W\supset\phi(\alpha')$, and interchanging α and α', we have $\phi(\alpha')+W\supset\phi(\alpha)$, and (ii) follows.

(ii)$\rightarrow$(iii). Trivial.

(iii)$\rightarrow$(i). Let $V\in\mathscr{A}$. We claim that continuity of ϕ at 1 forces $A+nV$ to contain B for some n, and continuity of ϕ at 0 forces $B+nV$ to contain A for some n. We establish only the first statement; the proof of the second is identical. Since ϕ is continuous at $\alpha=1$, there exists $\varepsilon>0$ for which $\phi(1)+V\supset\phi(1-\varepsilon)$, i.e., $A+V\supset(1-\varepsilon)A+\varepsilon B$. Suppose the inclusion $A+(2/\varepsilon)V\supset B$ fails. In this event, choose $b\in B\setminus(A+(2/\varepsilon)V)$. Since $A+(2/\varepsilon)V$ contains the closure of $A+(1/\varepsilon)V$ (see, e.g., p. 40 of [5]), we can strongly separate b from $\mathrm{cl}(A+(1/\varepsilon)V)$, there exists $f\in X^*$ with

$$\begin{aligned}f(b) &> \sup\left\{f(z): z\in A+\frac{1}{\varepsilon}V\right\}\\ &= \sup\{f(a): a\in A\}+\frac{1}{\varepsilon}\sup\{f(v): v\in V\}.\end{aligned}$$

Denote the excess of $f(b)$ over this supremum by λ. Choose $\delta>0$ satisfying $(1-\varepsilon)\delta<\varepsilon\lambda$, and select $a_0\in A$ with $f(a_0)>\sup\{f(a): a\in A\}-\delta$. Then $(1-\varepsilon)a_0+\varepsilon b\in(1-\varepsilon)A+\varepsilon B$, and

$$\begin{aligned}f((1-\varepsilon)a_0+\varepsilon b) &= (1-\varepsilon)f(a_0)+\varepsilon\lambda+\varepsilon\sup\{f(a): a\in A\}+\sup\{f(v): v\in V\}\\ &> (1-\varepsilon)(\sup\{f(a): a\in A\}-\delta)+\varepsilon\lambda\\ &\quad +\varepsilon\sup\{f(a): a\in A\}+\sup\{f(v): v\in V\}\\ &> \sup\{f(z): z\in A+V\}.\end{aligned}$$

As a result, $(1-\varepsilon)a_0+\varepsilon b$ fails to lie in $A+V$, contradicting $\phi(1)+V\supset$

$\phi(1-\varepsilon)$. We conclude that $A+(2/\varepsilon)V \supset B$, and since V is balanced, $A+nV \supset B$ for each $n > 2/\varepsilon$.

(ii)$\rightarrow$(iv). Trivial.

(iv)$\rightarrow$(v). Trivial.

(v)$\rightarrow$(i). Suppose (i) fails for some $V \in \mathscr{A}$. Using the fact that $V+nV=(n+1)V$ for each positive integer n, it is a routine matter to show that

$$\{C \in C(X) : A+nV \supset C \quad \text{and} \quad C+nV \supset A \quad \text{for some} \quad n \in Z^+\}$$

is both open and closed in $\langle C(X), \tau_H \rangle$. We conclude that A and B are not connected in the hyperspace.

A number of remarks are in order. First, the full strength of continuity of ϕ at $\alpha=0$ and $\alpha=1$ was not used in the proof of (iii)$\rightarrow$(i), only the so-called *Hausdorff upper semicontinuity* of ϕ, a property that has proved important in the approximation of multifunctions (see, e.g., [2] or [3]). Evidently, even without the convexity of A and B, the map ϕ, viewed as a multifunction, is *lower semicontinuous in the sense of Kuratowski* (see, e.g., [7] or [8]), i.e., whenever W is open in X, then $\{\alpha : \phi(\alpha) \cap W \neq \emptyset\}$ is open in $[0,1]$. Also, the necessity of (i) for connectedness of A and B does not depend on the convexity of A and B. Most importantly, when X is a normed linear space, condition (i) simply says that the Hausdorff distance from A to B is finite.

As one might expect, the straight line function for nonconvex closed sets remains continuous provided both sets are bounded in the locally convex space. This is established by the technique of proof of (i)$\rightarrow$(ii), in conjunction with the following lemma.

LEMMA. *Let A be a bounded set in a locally convex space X, and let $W \in \mathscr{A}$. Then there exists $\varepsilon > 0$ such that whenever α and α' are positive numbers with $\alpha - \varepsilon < \alpha' < \alpha$, we have $\alpha A + W \supset \alpha' A + (\alpha - \alpha')A$.*

PROOF. Choose $V \in \mathscr{A}$ with $V + V \subset W$. Since A is bounded and V is balanced, there exists $\varepsilon > 0$ such that if $0 \leqq |\lambda| < \varepsilon$, then $\lambda A \subset V$. Suppose α' and α are as in the statement of the lemma, and $x \in \alpha' A$ and $y \in (\alpha - \alpha')A$ are arbitrary. Choose $a \in A$ with $x = \alpha' a$. We have

$$x + y = \alpha a + (\alpha' - \alpha)a + y \in \alpha A + V + V \subset \alpha A + W.$$

Now if A is a closed and bounded set and B is a closed set satisfying condition (i) of Theorem 1 with respect to A, then B must also be bounded. Combining the above facts, we have the following theorem.

THEOREM 2. *Let X be a locally convex space. Then the closed, bounded convex sets form both a component and a path component of $\langle C(X), \tau_H \rangle$,, and the closed and bounded sets form both a component and a path component of the space of all nonempty closed subsets of X topologized by τ_H.*

It would be useful to present an even more explicit condition equivalent to condition (i). An important necessary condition for closed convex sets A and B to satisfy (i) is that they have the same directions of recession.

DEFINITION. A vector z in a linear space X is called a *direction of recession* for a given convex set A if for each $a \in A$, $a + z$ is again in A.

A vector z will be a direction of recession for a closed convex set A if and only if for some $a_0 \in A$, $\{a_0 + \lambda z : \lambda \geqq 0\} \subset A$. It is well known that the set of such directions z forms a closed convex cone, which we denote by $0^+(A)$, because $0^+(A) = \bigcap \{\lambda A : \lambda > 0\}$ whenever the origin lies in A [10].

THEOREM 3. *Let A and B be closed convex subsets of a locally convex space X which are path connected in $\langle C(X), \tau_H \rangle$. Then $0^+(A) = 0^+(B)$.*

PROOF. Since translation is continuous in $\langle C(X), \tau_H \rangle$ and recession cones are not altered by translations, we may assume that both A and B contain the origin θ. Suppose a is a nonzero direction of recession for A, i.e., $\{\lambda a : \lambda \geqq 0\} \subset A$. It suffices to show that for each $\lambda > 0$, the point λa lies in B. Let $V \in \mathscr{A}$ be arbitrary, and choose $n \in Z^+$ with $B + nV \supset A$. Since $(n\lambda)a \in A$, there exists $v \in V$ and $b \in B$ with $(n\lambda)a = b + nv$. Since B is convex and contains θ, we have $\lambda a = (1/n)b + v \in B + V$, so that $B \cap (\lambda a + V) \neq \emptyset$. Thus, $\lambda a \in \mathrm{cl}(B) = B$.

Another fundamental topology on the closed subsets of a space is the *finite* or *Vietoris topology* τ_V which has as subbasic open sets all sets of the form

$$V^- = \{F : F \text{ is closed and } F \cap V \neq \emptyset\},$$
$$V^+ = \{F : F \text{ is closed and } F \subset V\},$$

where V runs over the open subsets of the space. Basic facts about this topology can be found in the fundamental paper of Michael [9] or the recent monograph of Klein and Thompson [7].

If X is a locally convex space then it follows rather easily from known facts that $\langle C(X), \tau_V \rangle$ is connected. To see this, by Theorem 2.4.4 of [7], the hyperspace of all finite subsets of X is connected in τ_V. Let us denote the set of all such finite subsets by $F(X)$. We claim that the convex hull operator

$$\mathrm{conv}:\langle F(X),\tau_V\rangle\to\langle C(X),\tau_V\rangle$$

is continuous. We show that the inverse image of each subbasic open set is open. First, suppose that $\mathrm{conv}(\{x_1,\dots,x_n\})\in V^-$ for some open subset V of X. This means that $\mathrm{conv}(\{x_1,\dots,x_n\})\cap V\neq\emptyset$. By the continuity of the operations in X and since V is open, there exists for each $i\in\{1,\dots,n\}$ an open convex neighborhood U_i of x_i and $\alpha_i>0$ with $\sum\alpha_i=1$ such that $\sum\alpha_i z_i\in V$ whenever $z_i\in U_i$ for each i. Since each U_i is convex, we have

$$\{x_1,\dots,x_n\}\in\left[\bigcap_{i=1}^n U_i^-\right]\cap\left[\bigcup_{i=1}^n U_i\right]^+\cap F(X)\subset\mathrm{conv}^{-1}(V^-),$$

so the inverse image of V^- is open in $F(X)$. On the other hand, if $\mathrm{conv}(\{x_1,\dots,x_n\})\in V^+$, i.e., $\mathrm{conv}(\{x_1,\dots,x_n\})\subset V$, and for each $W\in\mathscr{A}$ there exists $z_i(W)\in x_i+W$ for $i=1,\dots,n$ for which

$$\mathrm{conv}(\{z_1(W),z_2(W),\dots,(W)\})\cap V^c\neq 0,$$

then routine compactness arguments will yield $\mathrm{conv}(\{x_1,\dots,x_n\})\cap V^c\neq\emptyset$, an impossibility. As a result there exists $W\in\mathscr{A}$ for which

$$\{x_1,\dots,x_n\}\in\left[\bigcap_{i=1}^n (x_i+W)^-\right]\cap\left[\bigcup_{i=1}^n (x_i+W)\right]^+\cap F(X)\subset\mathrm{conv}^{-1}(V^+)$$

so the inverse image of V^+ is also open in $F(X)$. This establishes the continuity of the convex hull operator, and since the continuous image of a connected space is connected, the space of convex polytopes with the Vietoris topology must be connected. The τ_V-density of the polytopes in $C(X)$ is obvious, and we conclude that $\langle C(X),\tau_V\rangle$ is connected.

Path connectedness in $\langle C(X),\tau_V\rangle$, on the other hand, seems rather rare, or at least hard to determine. Each pair of compact convex sets can be joined continuously by the straight line path between them because $\tau_V=\tau_H$ on the compact sets (see, e.g., page 46 of [1]). But bounded convex sets cannot in general be so connected, even if one is a deformation retract of the other. For example, let $\{e_n : n\in Z^+\}$ be the standard orthonormal base in l_2, and let $A=U_0$ and let $B=2U_0$ in l_2. Then

$$\phi(\alpha)=\mathrm{cl}\,(\alpha U_0+(1-\alpha)(2U_0))=(2-\alpha)U_0$$

is not continuous with respect to the Vietoris topology at $\alpha=1$ because if

$$V=\left\{\frac{n+1}{n}\,e_n : n\in Z^+\right\}^c,$$

then $\phi(1)\in V^+$, but for each $\alpha<1$, $\phi(\alpha)\notin V^+$.

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